

Stochastic thermodynamics of macrospins with fluctuating amplitude and direction

Swarnali Bandopadhyay,^{1,*} Debasish Chaudhuri,^{2,†} and A. M. Jayannavar^{3,‡}

¹*TIFR Centre for Interdisciplinary Sciences, 21 Brundavan Colony, Narsingi, Hyderabad 500075, Telengana, India*

²*Indian Institute of Technology Hyderabad, Yeddumailaram 502205, Telengana, India*

³*Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, India.*

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We consider stochastic energy balance and entropy production (EP) in a generalized Langevin dynamics of macrospins, allowing for both amplitude and direction fluctuations, under external magnetic field. EP is calculated using Fokker-Planck equation, distinguishing between reversible and irreversible parts of probability currents. The system entropy increases due to irreversible non-equilibrium processes, and reduces as heat dissipates to surrounding environment. Using path probability distributions of time-forward trajectories and conjugate trajectories under time reversal, we obtain fluctuation theorems (FT) for total stochastic EP. We show that the choice of conjugate trajectories is crucial in obtaining entropy like quantities that obey FTs.

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I. INTRODUCTION

Stochastic spin dynamics under magnetic fields and the influence of other spins, plays an important role in understanding magnetic properties of condensed matter systems. With the advent in nano-technology, the size of magnetic devices like magnetic read head and random access memory are being reduced consistently. This makes them vulnerable to thermal fluctuations [1, 2]. Understanding the role of stochasticity in such devices is thus becoming important, even from the perspective of better control of their performance [3–7]. The classical dynamics of a magnetization \mathbf{m} under external field \mathbf{H} is described by the Heisenberg motion $\dot{\mathbf{m}} = \gamma \mathbf{m} \times \mathbf{H}$ where γ denotes the gyromagnetic ratio [8]. This dynamics, evidently, conserves the amplitude $m = |\mathbf{m}|$. When coupled to a heat bath, the dynamics gets stochastic and is often expressed as a stochastic Landau-Lifshitz (sLL) equation [9]

$$\dot{\mathbf{m}} = \gamma \mathbf{m} \times [(\mathbf{H} + \mathbf{h}(t)) - \eta'(\mathbf{m} \times \mathbf{H})]. \quad (1)$$

Kubo and Hashitsume argued for the introduction of dissipation term $-\eta' \mathbf{m} \times (\mathbf{m} \times \mathbf{H})$ along with the stochastic fluctuation $\mathbf{h}(t)$ to obtain fluctuation-dissipation relation [9]. Here $\mathbf{h}(t)$ is regarded as a Gaussian white noise with $\langle \mathbf{h}(t) \rangle = 0$, and $\langle \mathbf{h}(t) \otimes \mathbf{h}(t') \rangle = 2D'_0 \mathbf{1} \delta(t - t')$ with $\mathbf{1}$ denoting an identity matrix, $D'_0 = \eta' k_B T / V$ where T is the temperature, V the physical volume of the macrospin, k_B Boltzmann constant. The sLL equation was independently derived using the Zwanzig formalism of coupling the spin dynamics with harmonic bath and taking the Markovian limit [10]. This equation describes a stochastic rotational dynamics of magnetization, keeping

the magnitude m conserved. In Ref. [11], the properties of Fokker-Planck equation corresponding to a related Landau-Lifshitz-Gilbert (LLG) equation were analyzed in detail. The constant m dynamics is a good approximation for bulk ferromagnets at room temperature, where the transition temperatures for ferromagnetic to paramagnetic phase transitions are much larger. Stochastic fluctuations of m occur within a ferromagnetic domain due to exchange interaction, with enhanced effect near the transition temperature [8]. The fluctuations in m becomes dominant in bulk ferromagnets only at high temperatures. On the other hand, due to enhanced relative fluctuations in small ferromagnetic domains, e.g., in a macrospin, the transition temperatures get largely suppressed with reduction of system size [12, 13], enhancing the fluctuations in m even at room temperature. Recently, a generalized Langevin spin dynamics has been proposed that captures longitudinal fluctuations in the spin magnitude, as well as the stochastic rotation dynamics of its orientation [14]. In this paper, we present stochastic thermodynamics of a macrospin system, deriving stochastic energy balance relation and fluctuation theorems for probability of entropy production.

During the last two decades, a theoretical description of stochastic thermodynamics has been developed to describe non-equilibrium small systems having enhanced relative fluctuations, using stochastic counterparts of thermodynamic variables like energy, work, entropy etc. [15–22]. While the possibility of second law violating stochastic trajectories was recognized long back [23], it took several decades before it was shown that probabilities of such trajectories in steady state $P(-\Delta s_t)$, with $-\Delta s_t$ denoting negative entropy production, are exponentially suppressed with respect to the positive entropy producing ones via the relation $P(\Delta s_t)/P(-\Delta s_t) = \exp(\Delta s_t/k_B)$ [24–26]. This relation is known as the detailed fluctuation theorem. This, and a related integral fluctuation theorem $\langle \exp(-\Delta s_t/k_B) \rangle = 1$, which is equivalent to the Jarzynski equality for non-equilibrium transformations from an initial equilibrium

*Electronic address: swarnalib@tifrh.res.in

†Electronic address: debc@iith.ac.in

‡Electronic address: jayan@iopb.res.in

state to a final state that eventually reaches equilibrium, have been derived [16–18, 27–30]. These theorems were verified using experiments on colloids [31–33], granular matter [34, 35], and used to obtain the free energy landscape of RNA [36, 37], and torque produced by F1-ATPase motor proteins [38].

In the following section, we present the generalized Langevin dynamics of macrospins (GLDM). We discuss its motivation, and corresponding projected dynamics along the longitudinal and transverse directions. Next, we study its stochastic thermodynamics, first deriving the stochastic energy balance, and then entropy production using Fokker-Planck equation and ratio of time-forward and conjugate path-probabilities. Our analysis shows that it is possible to obtain fluctuation theorems for entropy like quantities, each of which emerges out of a specific way of choosing conjugate trajectories. The time-reversed trajectories give fluctuation theorems in terms of EP in reservoir given by the dissipated heat, which is consistent with the results of Fokker-Planck equation. Another possible choice of conjugate trajectories leads to an entropy like quantity that also involves gyroscopic work done due to magnetic field induced spin torque. This quantity also obeys both detailed and integral fluctuation theorems. We present discussions interpreting our results. Finally, we conclude by presenting a summary.

II. MODEL

Consider a macrospin having magnetization \mathbf{m} , and volume V . The GLDM for the macrospin in presence of a time-dependent external magnetic field $\mathbf{H}(t)$ can be written as [14]

$$\dot{\mathbf{m}} = \gamma \left[\mathbf{m} \times \mathbf{H}(t) - \eta \frac{\partial g}{\partial \mathbf{m}} + \mathbf{h}(t) \right], \quad (2)$$

where $\dot{\mathbf{m}} \equiv d\mathbf{m}/dt$. The Langevin heat bath is characterized by the dissipation coefficient η , and the Gaussian white noise \mathbf{h} the components of which obey $\langle \mathbf{h}(t) \rangle = 0$, $\langle \mathbf{h}(t) \otimes \mathbf{h}(t') \rangle = 2D_0 \mathbf{1} \delta(t - t')$ with $D_0 = \eta k_B T / V$. In the above equation $\mathbf{m} \times \mathbf{H}$ denotes a non-conservative spin-torque. The energy density is given by

$$g = (f_L - \mathbf{m} \cdot \mathbf{H}(t))$$

where, $f_L = -\frac{a}{2}m^2 + \frac{b}{4}m^4$, (3)

is the Landau free energy density, and the effective magnetic field $\mathbf{H}^{\text{eff}} = -\partial g / \partial \mathbf{m} = \mathbf{H}^{\text{int}} + \mathbf{H}(t)$, with $\mathbf{H}^{\text{int}} = (a - bm^2)\mathbf{m}$ being the mean field contribution due to collective spin alignment. g can be expressed as $g = -\mathbf{m} \cdot \mathbf{H}^{\text{eff}}$. f_L denotes the Landau free energy density having two equivalent minima at $\mathbf{m} = \pm \sqrt{a/b} \hat{m}$, with $a = a_0(T_c - T) > 0$ in the ferromagnetic phase, where T_c is the transition temperature, and T is the temperature of the system [8]. With reduction of system size,

T_c decreases. It was shown for three dimensional Ising clusters with total number of spins N , the transition temperature decreases with reduction in macrospin size N as $T_c \sim T_c^\infty (1 - 1/N^\phi)$ where $\phi \approx 1/3$, and T_c^∞ denotes the transition temperature of thermodynamically large system [13]. Thus for small enough size of a macrospin, T_c approaches T from above, thereby increasing fluctuations in m . The term $-\mathbf{m} \cdot \mathbf{H}$ in g is due to external magnetic field \mathbf{H} , and shifts the global minimum towards positive \hat{m} . Thus the GLDM may be expressed as

$$\dot{\mathbf{m}} = [\mathbf{m} \times \mathbf{H}(t) + \eta \mathbf{H}^{\text{eff}}(t) + \mathbf{h}(t)], \quad (4)$$

absorbing γ into the definition of time, $t \rightarrow \gamma t$.

Eq. (2) may be motivated by drawing parallel to Langevin equations of motion of driven diffusing particles [14]. Note that such a particle in one dimension obeys $\dot{v} = f(t) - \eta v + \xi$, where v denotes the particle velocity, $f(t)$ a time-dependent external force. The viscous dissipation $-\eta v$ and the Gaussian white noise ξ are forces due to coupling to the heat bath, with $\langle \xi(t) \rangle = 0$, and $\langle \xi(t)\xi(t') \rangle = 2\eta k_B T \delta(t - t')$. The viscous dissipation $-\eta v$ may be rewritten as $-\eta \partial_v \mathcal{H}$ given that the velocity dependence of the Hamiltonian \mathcal{H} is $v^2/2$. Using this as a guiding principle, one may replace v by the magnetic moment \mathbf{m} , as both are odd parity variables under time reversal. Similarly, the force f may be replaced by the external torque due to the magnetic field $\mathbf{m} \times \mathbf{H}$. Replacing Hamiltonian \mathcal{H} by energy density $g = -\mathbf{m} \cdot \mathbf{H}$ for a single spin, and $\partial_v \mathcal{H}$ by $\partial g / \partial \mathbf{m} = -\mathbf{H}$ we obtain the GLDM for a single spin $\dot{\mathbf{m}} = [\mathbf{m} \times \mathbf{H} + \eta \mathbf{H} + \mathbf{h}(t)]$. Note that in this equation, the term $\eta \mathbf{H} + \mathbf{h}$ denotes the force and torque due to the heat bath [14]. Extending this argument to a macrospin containing large number of spins, one obtains Eq.(4) by using $g = -\mathbf{m} \cdot \mathbf{H} + f_L$. In Eq.(4), the term $\eta \mathbf{H}^{\text{eff}} + \mathbf{h}$ denotes the force and torque on the macrospin due to the heat bath. Throughout this paper, we use Stratonovich convention while interpreting stochastic differential equations.

It is possible to separate the longitudinal and transverse dynamics of the macrospin \mathbf{m} . Taking longitudinal projection, i.e., projecting Eq. (2) along $\hat{m} = \mathbf{m}/m$ we obtain the dynamics for the spin amplitude

$$\dot{m} = [\eta H_{\parallel}(t) + h_{\parallel}(t) + \eta(am - bm^3)] , \quad (5)$$

where $H_{\parallel}(t) = \hat{m} \cdot \mathbf{H}(t)$ and $h_{\parallel} = \hat{m} \cdot \mathbf{h}$, with $\langle h_{\parallel}(t) \rangle = 0$, and $\langle h_{\parallel}(t)h_{\parallel}(t') \rangle = 2D_0 \delta(t - t')$. Clearly, in this equation, $\eta(H_{\parallel} + am - bm^3) + h_{\parallel}$ is the longitudinal force due to the Langevin heat bath. The corresponding Fokker-Planck equation is $\partial_t P(m, t) = -\partial_m j$ where $j = -D_0 \partial_m P + \eta[am - bm^3 + H_{\parallel}]$. For a time-independent magnetic field, setting the dissipative current $j = 0$, one obtains the detailed balanced equilibrium distribution, $P_{\text{eq}}(m) = P_0 \exp(-g_{\parallel} V / k_B T)$ with the energy density $g_{\parallel} = f_L - H_{\parallel} m$.

Subtracting the longitudinal dynamics Eq.(5) from Eq.(4), one obtains

$$\dot{\mathbf{m}}_{\perp} = \mathbf{m} \times \mathbf{H}(t) + \eta \mathbf{H}_{\perp}(t) + \mathbf{h}_{\perp}(t), \quad (6)$$

where $\mathbf{m}_\perp = \mathbf{m} - \hat{m}\mathbf{m}$, $\mathbf{H}_\perp = \mathbf{H} - \hat{m}H_\parallel = -\hat{m} \times (\hat{m} \times \mathbf{H})$, and $\mathbf{h}_\perp = \mathbf{h} - \hat{m}h_\parallel$. It can be shown that \mathbf{h}_\perp and $\tilde{\mathbf{h}}_\perp = \hat{m} \times \mathbf{h}$ obeys the same statistics : $\langle \mathbf{h}_\perp \rangle = 0 = \langle \tilde{\mathbf{h}}_\perp \rangle$, $\langle \mathbf{h}_\perp(t) \otimes \mathbf{h}_\perp(t') \rangle = (\mathbf{1} - \hat{m} \otimes \hat{m})\delta(t - t') = \langle \tilde{\mathbf{h}}_\perp(t) \otimes \tilde{\mathbf{h}}_\perp(t') \rangle$ [14]. Thus one can replace \mathbf{h}_\perp by $\tilde{\mathbf{h}}_\perp$ in Eq.(6). The resultant equation can be expressed as

$$\dot{\mathbf{m}}_\perp = \mathbf{m} \times (\mathbf{H}(t) + \mathbf{h}'(t)) - \eta' \mathbf{m} \times (\mathbf{m} \times \mathbf{H}(t)), \quad (7)$$

where $\mathbf{h}' = \mathbf{h}/m$, and $\eta' = \eta/m^2$. Note that for constant magnitude m , $\dot{\mathbf{m}}_\perp = \dot{\mathbf{m}}$, and Eq.(7) is then same as the sLL equation Eq.(1). The term $-\eta' \mathbf{m} \times (\mathbf{m} \times \mathbf{H})$ denotes the Gilbert damping, and $\mathbf{m} \times \mathbf{h}'$ denotes the stochastic part of the total torque imparted due to the Langevin heat bath.

III. RESULTS AND DISCUSSIONS

A. Energy conservation

The macrospin undergoes a relaxation dynamics in the Langevin heat bath, settling into an average unidirectional precession around the field \mathbf{H} . Unlike sLL equation which describes motion of spin constrained to have a constant magnitude, here amplitude $m = |\mathbf{m}|$ is not conserved, and obeys the above mentioned distribution $P_{\text{eq}}(m)$. The rate of change of energy density \dot{q} of the macrospin is given by $\dot{q} = -\dot{\mathbf{m}} \cdot \mathbf{H}^{\text{eff}} - \mathbf{m} \cdot \dot{\mathbf{H}}$, with $\mathbf{H}^{\text{eff}} = \mathbf{H} + H_{\text{int}}$ and $\mathbf{H}_{\text{int}} = -\partial f_L / \partial \mathbf{m}$. Note that \mathbf{H} and \mathbf{H}_{int} shares the same symmetry under time reversal, as does \mathbf{H} and \mathbf{m} . Substituting Eq. (2) in the expression of \dot{q} we obtain stochastic energy balance,

$$\dot{q} = \dot{q} + \dot{w}, \quad (8)$$

$$\text{where } \dot{q} = -\dot{\mathbf{m}} \cdot \mathbf{H}^{\text{eff}} \quad (9)$$

$$\text{and } \dot{w} = -\mathbf{m} \cdot \dot{\mathbf{H}}. \quad (10)$$

We used the sign convention that stochastic heat and work done are positive if they increase the energy of the system. Note that the stochastic energy balance presented above, is a relation between energy density, work density and heat absorption per unit volume. Here \dot{q} and \dot{w} represent the rate of heat absorbed by the system and the rate of work done on the system respectively. Note that for driven diffusive particles, heat absorbed by the system is given by $v(-\eta v + \xi)$, and we motivated the GLDM equation by replacing v by $-\mathbf{H}$, and ξ by \mathbf{h} . Thus it is only natural to identify $-(\eta H^2 + \mathbf{h} \cdot \mathbf{H})$ as heat absorbed by a single spin. For a system of spins present in the macrospin, \mathbf{H} has to be replaced by \mathbf{H}^{eff} . Thus $\dot{q} = -[\eta(H^{\text{eff}})^2 + \mathbf{h} \cdot \mathbf{H}^{\text{eff}}] = -\dot{\mathbf{m}} \cdot \mathbf{H}^{\text{eff}}$. This suggests a stochastic version of Clausius entropy production in the heat bath in the form $-\dot{q}/T$. In the following, we present a careful analysis of entropy production.

B. Entropy production and heat dissipation

With $P(\mathbf{m}, t)$ denoting the probability of finding a spin in the state \mathbf{m} at time t , the non-equilibrium Gibbs entropy $S = -k_B \int d\mathbf{m} P \ln P(\mathbf{m}, t)$ suggests a definition of time dependent stochastic entropy of the system $s(t) = -k_B \ln P(\mathbf{m}, t)$ where $S = \langle s \rangle$ denotes the ensemble average of stochastic entropy [30]. Note that, here and in the rest of the paper, whenever we mention entropy, energy or work done, it actually means the quantity *per unit volume*. To achieve this in the following, we replace diffusivity of magnetization D_0 by $D = \eta k_B T$.

The rate of change in stochastic entropy is given by

$$\frac{\dot{s}}{k_B} = -\frac{\partial_t P}{P} - \frac{\partial_{\mathbf{m}} P}{P} \cdot \dot{\mathbf{m}}. \quad (11)$$

where the probability density $P(\mathbf{m}, t)$ obeys the Fokker-Planck equation $\partial_t P = -\partial_{\mathbf{m}} \cdot \mathbf{J}$ with $\partial_{\mathbf{m}} \equiv (\partial_{m_x}, \partial_{m_y}, \partial_{m_z})$ and probability flux \mathbf{J} . Note that under time reversal t , \mathbf{m} and \mathbf{H} change sign. Thus $\mathbf{J} = \mathbf{J}^{\text{rev}} + \mathbf{J}^{\text{irr}}$, where \mathbf{J}^{rev} is the reversible current that does not change sign under time reversal, and \mathbf{J}^{irr} is the irreversible current that changes sign [41]. The i -th component of these currents are given by

$$\begin{aligned} J_i^{\text{rev}} &= N_i P \\ J_i^{\text{irr}} &= \eta H_i^{\text{eff}} P - D \partial_{m_i} P, \end{aligned} \quad (12)$$

where, the component of spin-torque due to external magnetic field $N_i = (\mathbf{m} \times \mathbf{H})_i$. Using Eq. (12) in Eq. (11) to replace $\partial_{\mathbf{m}} P$, one can express the rate of change in stochastic entropy as

$$\begin{aligned} \frac{\dot{s}}{k_B} &= -\frac{\partial_t P}{P} + \frac{J_i^{\text{irr}} \dot{m}_i}{PD} - \frac{\eta}{D} H_i^{\text{eff}} \dot{m}_i \\ &= -\frac{\partial_t P}{P} + \frac{J_i^{\text{irr}} \dot{m}_i}{PD} - \frac{1}{k_B T} \dot{q}. \end{aligned} \quad (13)$$

At this stage, let us perform a two step averaging, (i) over trajectories, and (ii) over phase space by integrating over all \mathbf{m} with probability $P(\mathbf{m}, t)$. The trajectory average of the components of magnetization dynamics depends on both reversible and irreversible parts of probability flux, $\langle \dot{m}_i | \mathbf{m}, t \rangle = J_i / P = (J_i^{\text{rev}} + J_i^{\text{irr}}) / P = N_i + J_i^{\text{irr}} / P$ [30]. Thus after the trajectory average one can replace $(J_i^{\text{irr}} \dot{m}_i) / (PD)$ by $[(J_i^{\text{irr}} N_i) / (PD) + (J_i^{\text{irr}})^2 / (P^2 D)]$. Now let us perform averaging over the probability density $P(\mathbf{m}, t)$ by multiplying Eq.(13) throughout by $P(\mathbf{m}, t)$ and performing integration over \mathbf{m} . The conservation of probability $\int d\mathbf{m} P = 1$ leads to $\int d\mathbf{m} \partial_t P = 0$. Thus one obtains the final average

$$\frac{\dot{S}}{k_B} = \frac{\langle \dot{s} \rangle}{k_B} = \frac{1}{D} \int d\mathbf{m} \frac{(J_i^{\text{irr}})^2}{P} + \frac{1}{D} \int d\mathbf{m} J_i^{\text{irr}} N_i - \frac{\langle \dot{q} \rangle}{k_B T}.$$

Now using the expression in Eq.(12) one can show that the second term in the above equation $\int d\mathbf{m} J_i^{\text{irr}} N_i = 0$.

This term vanishes, as (i) $H_i^{\text{eff}} N_i = 0$ due to vector identities $\mathbf{H} \cdot (\mathbf{m} \times \mathbf{H}) = 0$ and $\mathbf{m} \cdot (\mathbf{m} \times \mathbf{H}) = 0$, (ii) $\int d\mathbf{m} N_i \partial_{m_i} P = 0$ using integration by parts. Thus

$$\dot{S} = \langle \dot{s} \rangle = \frac{1}{\eta T} \int d\mathbf{m} \frac{(J_i^{\text{irr}})^2}{P} - \frac{\langle \dot{q} \rangle}{T} \equiv \Pi - \Phi, \quad (14)$$

where $\Pi = \frac{1}{\eta T} \int d\mathbf{m} \frac{(J_i^{\text{irr}})^2}{P}$ is the EP in the system due to irreversible processes quantified by J_i^{irr} , and $\Phi = \langle \dot{q} \rangle / T$ is the entropy flux to the reservoir due to average heat loss. At this stage, it is interesting to note that as the system gets isolated from the heat bath, i.e., $\eta \rightarrow 0$, $\Pi \sim \eta \rightarrow 0$, a result expected for EP in an isolated system. The total EP in the combined system and reservoir obeys the second law of thermodynamics, $\dot{S}_t = \dot{S} + \Phi = \frac{1}{\eta T} \int d\mathbf{m} \frac{(J_i^{\text{irr}})^2}{P} \geq 0$, where the equality denotes equilibrium with $J_i^{\text{irr}} = 0$. At steady state, $\Phi = \Pi = \langle \dot{q} \rangle / T$, and average change in energy $\langle \dot{q} \rangle = 0$ leads to $\langle \dot{q} \rangle = -\langle \dot{w} \rangle = \langle \mathbf{m} \cdot \dot{\mathbf{H}} \rangle$. Thus one can express the average entropy flux as $\Phi = \langle \mathbf{m} \cdot \dot{\mathbf{H}} \rangle / T$. The steady state EP in the reservoir is due to non-equilibrium processes driven by time-dependent external field $\mathbf{H}(t)$.

The above discussion shows that the stochastic EP in the reservoir is

$$\dot{s}_r = -\frac{\dot{q}}{T}. \quad (15)$$

This quantity can be both positive or negative. Oono and Paniconi [40] introduced a concept of housekeeping heat, which is the heat dissipated to keep the system at non-equilibrium steady state. As we have seen above, at steady state, the average heat dissipated is equal to the mean work done by the system $\langle -\mathbf{m} \cdot \dot{\mathbf{H}} \rangle$. Thus the expression of stochastic housekeeping heat generation $\dot{q}_h = -\mathbf{m} \cdot \dot{\mathbf{H}}$. This gives the rate of excess heat generation $\dot{q}_e = \dot{q} - \dot{q}_h = \mathbf{m} \cdot \dot{\mathbf{H}} - \dot{\mathbf{m}} \cdot \mathbf{H}^{\text{eff}}$. If one changes the magnetic field from some initial value to a final value, the average excess heat generation remains non-zero transiently before the system relaxes from one steady state to another.

C. Equilibrium detailed balance

The steady state condition is given by $\partial_{m_i} [J_i^{\text{rev}} + J_i^{\text{irr}}] = 0$. At equilibrium, the dissipative current must vanish, $J_i^{\text{irr}} = 0$. This leads to the condition $dP/P = \beta H_i^{\text{eff}} dm_i$. The equation can be integrated for a time-independent external field \mathbf{H} to give

$$P = P_0 \exp[-\beta \{f_L - \mathbf{m} \cdot \mathbf{H}\}],$$

where $f_L = -(a/2)m^2 + (b/4)m^4$. Using the relation $J_i^{\text{irr}} = 0$ in the steady state condition one obtains $\partial_{m_i} J_i^{\text{rev}} = 0$, which is readily obeyed. These two relations, $J_i^{\text{irr}} = 0$ and $\partial_{m_i} J_i^{\text{rev}} = 0$ define the equilibrium detailed balance condition. A time-dependent magnetic field brings the system out of equilibrium, and allows EP.

D. Entropy production using path probabilities: Fluctuation theorems

EP along stochastic trajectories of a non-equilibrium system may also be estimated by using the inequality of probabilities of time-forward trajectories, and conjugate trajectories under suitably time-reversed protocol. We consider the time evolution of a macrospin from $t = 0$ to τ_0 through a path $X = [\mathbf{m}(t), \mathbf{H}(t)]$ where $\mathbf{H}(t)$ acts as control parameter, the functional form of which gives a specific protocol. Let us divide the path into $i = 1, 2, \dots, N$ segments, each of time-interval δt such that $N\delta t = \tau_0$. The transition probability $p_i^+(\mathbf{m}', t + \delta t | \mathbf{m}, t)$ on i -th infinitesimal segment is governed by the Gaussian random noise \mathbf{h}_i at i -th instant obeying probability distribution $P(\mathbf{h}_i) = (\delta t / 4\pi D)^{1/2} \exp(-\delta t \mathbf{h}_i^2 / 4D)$. Denoting Eq.(2) as $\dot{\mathbf{m}} = \Phi(\mathbf{m}(t), \mathbf{H}(t))$, the transition probability on i -th segment $p_i^+ = \mathcal{J}_i^+ \int d\mathbf{h}_i P(\mathbf{h}_i) \delta(\dot{\mathbf{m}}_i - \Phi_i)$, where the Jacobian of transformation at i -th instant of time $\mathcal{J}_i^+ = \det[(\partial \mathbf{h} / \partial \mathbf{m})_i]$. Using Stratonovich discretization, one can show

$$\mathcal{J}_i^+ = \frac{1}{\delta t} \left[1 - \frac{\delta t}{2} \frac{\partial \mathcal{F}(\mathbf{m}_i)}{\partial \mathbf{m}_i} \right] \quad (16)$$

where $\mathcal{F}(\mathbf{m}_i) = (\mathbf{m} \times \mathbf{H})_i + \eta[(a - bm^2)\mathbf{m}]_i + \eta \mathbf{H}_i$. Note that $\partial(\mathbf{m} \times \mathbf{H})_i / \partial \mathbf{m}_i = 0$, and $\partial \mathbf{H}_i / \partial \mathbf{m}_i = 0$. Thus the operative part of $\mathcal{F}(\mathbf{m}_i)$ in the above relation is the effective field contribution $\mathbf{H}_i^{\text{int}} = \eta[(a - bm^2)\mathbf{m}]_i$. Eq.(16) can be expressed as

$$\mathcal{J}_i^+ = \frac{1}{\delta t} \left[1 - \frac{\delta t}{2} \frac{\partial \mathbf{H}_i^{\text{int}}(\mathbf{m}_i)}{\partial \mathbf{m}_i} \right] \quad (17)$$

The probability of a complete trajectory is $\mathcal{P}_+ = \prod_{i=1}^N p_i^+$.

Similarly, the conjugate trajectory under time-reversal may be discretized, and the probability of such complete trajectories may be expressed as $\mathcal{P}_- = \prod_{i=1}^N p_i^-$. There exists various possibilities to choose conjugate trajectories under time-reversed protocol [29, 42, 43]. The conjugate trajectory must be carefully chosen so that the ratio $\mathcal{P}_+ / \mathcal{P}_-$ serves as a measure of irreversibility of the process, and as a result characterizes EP in the surrounding environment.

Under time-reversal, \mathbf{H} and \mathbf{m} changes sign simultaneously. The corresponding conjugate trajectory is denoted by $X^\dagger = [-\mathbf{m}(\tau_0 - t), -\mathbf{H}(\tau_0 - t)]$. This is similar to requirement of reversal of external flow direction in Ref. [42], under time reversal. The probability of time-reversed trajectory $\mathcal{P}_- = \prod_{i=1}^N p_i^-$, where $p_i^- = \mathcal{J}_i^- \int d\mathbf{h}_i P(\mathbf{h}_i) \delta(\dot{\mathbf{m}}_i - \Phi_i(\tau_0 - t))$. It is easy to see from Eq.(17) that $\mathcal{J}_i^+ = \mathcal{J}_i^-$. After some algebra, one obtains the ratio of the two probabilities of forward and reverse paths $\frac{\mathcal{P}_+}{\mathcal{P}_-} = \exp(\Delta s_r / k_B)$, where

$$\frac{\Delta s_r}{k_B} = \frac{\eta}{D} \int_0^{\tau_0} dt \mathbf{H}^{\text{eff}} \cdot \dot{\mathbf{m}} = -\frac{\Delta q}{k_B T}. \quad (18)$$

Note that the expression of Δs_r presented above agrees with the EP given in Eq.(15). Let us now assume that s_0 and s_ℓ are stochastic entropies of the system corresponding to its initial and final steady states respectively. So, the change in stochastic system entropy is given by $\Delta s = s_\ell - s_0 = k_B \ln(P_0/P_\ell)$ where $P_0(\mathbf{m}_0, \mathbf{H}_0)$ and $P_\ell(\mathbf{m}_\ell, \mathbf{H}_\ell)$ are distribution functions of these microstates.

As we have shown above, the change in reservoir entropy depends on the trajectory and is given by $\Delta s_r = k_B \ln(\mathcal{P}_+/\mathcal{P}_-)$. Thus the total entropy change

$$\Delta s_t = k_B \ln \left(\frac{P_0 \mathcal{P}_+}{P_\ell \mathcal{P}_-} \right) = \Delta s + \Delta s_r. \quad (19)$$

This immediately implies an integral fluctuation theorem (IFT) $\langle e^{-\Delta s_t/k_B} \rangle = 1$ [29]. Note that in deriving IFT, $\sum_X \equiv \sum_X^\dagger$ is used, as the Jacobian of transformation from time-forward path X to time-reversed path X^\dagger is unity [41]. Further, in a steady state, the total entropy change along a time-forward path Δs_t^f is equal and opposite to that along the time-reversed path, $\Delta s_t^r(X^\dagger) = -\Delta s_t^f(X)$. Using this, and Eq.(19) one obtains the following detailed fluctuation theorem (DFT) [17, 21] for probability distribution of EP $\rho(\Delta s_t)$ as

$$\rho(\Delta s_t) = e^{\Delta s_t/k_B} \rho(-\Delta s_t). \quad (20)$$

Using the definition $\Delta s_r = -\Delta q/T$ the IFT $\langle \exp(-\Delta s_t/k_B) \rangle = 1$ can be expressed as

$$\langle \exp(\beta \Delta q - \Delta s/k_B) \rangle = 1. \quad (21)$$

This is equivalent to Jarzynski relation, for transformations between non-equilibrium steady states [18, 39]. Due to Jensen inequality, this implies $T \langle \Delta s \rangle \geq \langle \Delta q \rangle$. For an infinitesimally slow variation of $\mathbf{H}(t)$, the equality holds, i.e., the steady state change in system entropy can be evaluated in terms of $\langle \Delta q \rangle \approx \langle -\mathbf{m} \cdot \Delta \mathbf{H} \rangle$. For a time-independent external field, one reaches an equilibrium steady state with $\langle \Delta q \rangle = 0$, and $\langle \Delta s \rangle = 0$.

E. Other possibilities of conjugate trajectories

Let us now consider, three other possibilities of choosing conjugate trajectories, such that one obtains entropy like quantities that obey DFT [44]. First, assume conjugate trajectories such that time forward protocol of $\mathbf{H}(t)$ traces back itself under time reversal. The corresponding conjugate trajectory is denoted by $X^\dagger = [\mathbf{m}(\tau_0 - t), \mathbf{H}(\tau_0 - t)]$ where \mathbf{m} and \mathbf{H} do not change sign. The probability of such conjugate trajectories is denoted by $\mathcal{P}_-^{(1)}$. Then the ratio of probabilities of time-forward and conjugate trajectories is $\mathcal{P}_+/\mathcal{P}_-^{(1)} = \exp(\Delta s_r^{(1)}/k_B)$

where

$$\begin{aligned} \frac{\Delta s_r^{(1)}}{k_B} &= \frac{\eta}{D} \int_0^{\tau_0} dt \mathbf{H}^{\text{eff}} \cdot \dot{\mathbf{m}} + \frac{1}{D} \int_0^{\tau_0} dt \mathbf{N} \cdot \dot{\mathbf{m}} \\ &= \frac{1}{k_B} [\Delta s_r + \Delta s_{\text{gyro}}], \end{aligned} \quad (22)$$

where $\Delta s_{\text{gyro}} = \Delta w_{\text{gyro}}/T$ with $\Delta w_{\text{gyro}} = (1/\eta) \int dt \mathbf{N} \cdot \dot{\mathbf{m}}$ being the gyroscopic work done on the system due to spin torque. One obtains the DFT

$$\rho(\Delta s_t^{(1)}) = e^{\Delta s_t^{(1)}/k_B} \rho(-\Delta s_t^{(1)})$$

where $\Delta s_t^{(1)} = \Delta s + \Delta s_r^{(1)}$. Numerical simulation of macrospins with constant amplitude m has been used in Ref. [43] to obtain the probability distribution $\rho(\Delta s_t^{(1)})$, which obeys DFT. This form of DFT may be interpreted as follows. One can define $\Delta \tilde{s}_t = \Delta s - \Delta q/T$, and rewrite the DFT as,

$$\frac{\rho(\Delta \tilde{s}_t, \Delta w_{\text{gyro}})}{\rho(-\Delta \tilde{s}_t, -\Delta w_{\text{gyro}})} = e^{\frac{1}{k_B} (\Delta \tilde{s}_t + \frac{\Delta w_{\text{gyro}}}{T})}.$$

In a steady state, ignoring Δs with respect to $\Delta q/T$, this relation leads to

$$\frac{\rho(-\Delta q, \Delta w_{\text{gyro}})}{\rho(\Delta q, -\Delta w_{\text{gyro}})} = e^{-\beta(\Delta q - \Delta w_{\text{gyro}})}. \quad (23)$$

This equality is closely related to the fluctuation theorem for heat engines [45–47], and was used in Ref. [48] in the context of an isothermal engine absorbing heat Δq and performing work Δw_{gyro} via spin torque.

The Jacobian of transformation from time forward trajectory X and the conjugate trajectory X^\dagger is unity. This leads to the IFT $\langle e^{-\Delta s_t^{(1)}/k_B} \rangle = 1$, which by Jensen's inequality gives $\langle \Delta s_t^{(1)} \rangle \geq 0$, a result equivalent to the second law of thermodynamics. The IFT obtained from Eq.(23) has the form $\langle e^{-\beta(\Delta q - \Delta w_{\text{gyro}})} \rangle = 1$, which after Jensen's inequality gives $\langle \Delta w_{\text{gyro}} \rangle / \langle \Delta q \rangle \leq 1$, meaning average work does not exceed average heat. Note that the torque \mathbf{N} is associated with the reversible part of probability current $\mathbf{J}^{\text{rev}} = \mathbf{N}P$, and thus does not contribute to heat flux. However, it still contributes towards an entropy like term $\Delta s_r^{(1)}$ that gives total entropy $\Delta s_t^{(1)}$ obeying DFT and IFT.

Next we assume that \mathbf{m} alone changes sign along the conjugate trajectories so that they are described by $X^\dagger = [-\mathbf{m}(\tau_0 - t), \mathbf{H}(\tau_0 - t)]$. We denote the path probabilities along such conjugate trajectories by $\mathcal{P}_-^{(2)}$. Then the ratio $\mathcal{P}_+/\mathcal{P}_-^{(2)} = \exp(\Delta s_r^{(2)}/k_B)$ where

$$\frac{\Delta s_r^{(2)}}{k_B} = \frac{1}{D} \int_0^{\tau_0} dt \mathbf{N} \cdot \dot{\mathbf{m}} \equiv \frac{\Delta s_{\text{gyro}}}{k_B}. \quad (24)$$

Again, $\Delta s_t^{(2)} = \Delta s + \Delta s_r^{(2)}$ obeys the DFT. However, the Jacobian of transformation from X to X^\dagger is not unity,

and the IFT is not obeyed by this quantity. This is expected, as $\Delta s_r^{(2)}$ depends only on \mathbf{N} , which is associated with reversible probability current, and should not give rise to second law like inequality.

The third alternative is to consider conjugate trajectories in which \mathbf{H} alone changes sign, i.e., $X^\dagger = [\mathbf{m}(\tau_0 - t), -\mathbf{H}(\tau_0 - t)]$. Denoting the probability of conjugate trajectory $\mathcal{P}_-^{(3)}$, one obtains $\mathcal{P}_+/\mathcal{P}_-^{(3)} = 1$, i.e., the corresponding stochastic EP in the reservoir $\Delta s_r^{(3)} = 0$.

The EP in reservoir associated with dissipated heat $\Delta s_r = \Delta s_r^{(1)} - \Delta s_r^{(2)}$. Note that the amplitude of magnetization can be approximated to be constant, for samples with Curie temperature much larger than room temperature. In such cases, the stochastic Langevin dynamics can be described as diffusion of a particle under suitable torque due to external field [43]. In the spherical polar coordinates, macrospin orientation (θ, ϕ) may be treated as even functions under time reversal. As a result one obtains an expression of entropy, which is equivalent to $\Delta s_r^{(1)}$ involving a gyroscopic term Δs_{gyro} . The probability distribution of total EP $\rho(\Delta s_t^{(1)})$ has been shown to obey DFT. Of course, even within that restricted dynamics, if one considers $X^\dagger = [-\mathbf{m}(\tau_0 - t), -\mathbf{H}(\tau_0 - t)]$ as the conjugate trajectory, one obtains EP in the reservoir $\Delta s_r = -\Delta q/T$, as is shown in the appendix of Ref. [43].

Among all possible prescriptions for constructing stochastic trajectories, the definition of Δs_r in Eq.(18) obtained by tracing back the time-reversed trajectory directly utilizing reverse protocol of $\mathbf{H}(t)$, such that, $X^\dagger = [-\mathbf{m}(\tau_0 - t), -\mathbf{H}(\tau_0 - t)]$ leads to the expression \dot{s}_r in Eq.(15) obtained from Fokker-Planck equation. The ratio of probabilities of time forward, and time reversed trajectories gives unity in presence of time reversal symmetry. Thus any other value of this ratio gives a measure of breaking of time-reversal symmetry, and thus the EP. Note that the derivation of \dot{s}_r in Eq.(15) depends only on the dynamics, not on any particular definition

of conjugate trajectory. Such definitions were used as mathematical construct to derive fluctuation theorems.

IV. SUMMARY

We studied stochastic thermodynamics for a macrospin of fluctuating amplitude and direction of magnetization subjected to external magnetic field. We considered a generalized Langevin dynamics of macrospins, taking into account (i) a stochastic rotational dynamics of the magnetization and (ii) its longitudinal fluctuations, (iii) a mean field approximation of the interaction between spins within the macrospin cluster, and an external magnetic field. We obtained several possible fluctuation theorems for entropy-like quantities found from using different choices of conjugate trajectories under time reversal. Only one of the possible choices gave $\Delta s_r = -\Delta q/T$, the entropy production (EP) in the reservoir due to dissipated heat $-\Delta q$, that agrees with the expression one obtains from Fokker-Planck equation. A second entropy like quantity $\Delta s_t^{(1)} = \Delta s - \Delta q/T + \Delta w_{\text{gyro}}/T$, where Δw_{gyro} is the rotational work done on the macrospin due to magnetic field induced spin torque, also obeys fluctuation theorems. The heat dissipation and gyroscopic work done, can be measured separately in experiments on macrospins, and our predictions regarding fluctuation theorems can be tested.

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